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## OPTIMAL PORTFOLIO UNDER $VaR$ AND $ES$

An analysis of the dependence structure among certain European indices (FTSE100, CAC40, DAX30, ATX20, PX, BUX and BIST) has been conducted. The main features of the financial data were studied: asymmetry, fat-tailedness (leptokurtosis), variability and mutual dependence. We have fitted a regime switching copula based model including asymmetric and fat-tailed copulas. All the indices are left-skewed and fat-tailed. Large indices are more skewed and less fat-tailed. The findings suggest that size of a market has an influence on its properties. A particular behaviour of the Turkish market suggests the importance of geographical factors. It is also suggested that the maturity of a market is insignificant in the analysis. Another important conclusion drawn from our empirical investigation is that  $VaR$  is a less exact risk measure than  $ES$ . However, the dynamics of the temporal and statistical properties of both measures are similar.

**Keywords:** *value at risk, expected shortfall, interdependence, regime copulas, vine copula*

### 1. Introduction

Value at risk ( $VaR$ ) is a concise measure of risk developed in the late 1980 [21]. Another popular measure of risk is expected shortfall ( $ES$ ). This provides a risk statistic that gives information “beyond  $VaR$ ”. Generally speaking,  $VaR$  defines the threshold for minimum expected losses with a certain level of confidence, but it does not supply any additional information about the tail of the P&L (profit and loss) distribution.  $ES$  averages the P&L probable returns beyond the  $VaR$  threshold. Therefore, expected shortfall supplies (as compared relation to  $VaR$ ) additional information concerning the size of losses (see, e.g. [18]).

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Much attention in the financial literature has been paid to diversification opportunities in the light of the presence of the contagion effect on international financial markets. Diversification opportunities are highly dependent on the dependence structure of the available assets. Modern tools for capturing the structure of dependences are copulas [4, 6, 9, 16, 20, 22, 25, 27]. In order to reflect the dynamics of financial returns (and their risk exposure) regime switching models are widely applied in the financial literature [5, 17].

In order to check contagion effects on financial markets Rodriguez in [26] used a copula based model for Latin American and East Asian countries. His model included regime switches. He detected predictive power for international financial contagion. Okimoto in [24] applied a copula model with regime switching, taking into account the US and UK stock markets. He proved an asymmetric dependence between stock indices from these countries. Ning [23] checked the dependence of stock returns from North America and East Asia. She detected that asymmetric, dynamic tail dependence is higher intra-continent than across continents.

The application of standard copulas such as the multivariate Gaussian or Student- $t$  along with exchangeable Archimedean copulas is from a practical point of view not reasonable in higher dimensions because standard multivariate copulas suffer from rather inflexible structures. These problems can be solved by the application of the canonical vine (C-vine) and drawable vine (D-vine) copulas described in Chapter 2 of this paper (a detailed mathematical description is given in [1] and [10]). These copulas are able to reflect complex dependence structures based on the rich variety of bivariate copulas as building cells.

The main ideas concerning vine copulas are presented in the earliest contributions by Joe [13–15], Bedford and Cooke [2, 3] and Kurowicka and Cooke [19]. The form of vine copulas are cascades of bivariate copulas known as pair copulas. Such pair copula constructions decompose a multivariate probability density into bivariate copulas. The advantage is that each pair-copula can be chosen independently from the others. Such decomposition allows great flexibility in dependence modeling. A researcher can model not only asymmetries and tail dependence but also conditional dependence to build more compact and relevant models. Vines have the advantages of multivariate copula modeling, i.e. the separation of marginal and dependence modeling, and the flexibility of bivariate copulas. We stress that many vine structure specifications are possible.

Chollete et al. [7] applied general canonical vines in order to model the portfolios of international stock returns from the G5 and Latin America. They established that this model is better than dynamic Gaussian and Student- $t$  copulas. The model proved to be a good tool in modifying the value at risk ( $VaR$ ) for these international stock returns. The paper delivered evidence on significant asymmetric dependence in joint asset returns.

In a recent contribution, Chollete et al. [8] using simple Archimedean copulas for countries from G5, Latin America and East Asia found that dependence has increased over time. Moreover, the authors found evidence for asymmetric dependence or downside risk in Latin America, but less in the G5. The results reflected very little downside risk in East Asia. Additionally, East Asian and Latin American returns exhibited correlation complexity. The regions with the maximal dependence or the worst diversification did not receive large returns. The results presented by the contributors suggest international limits to diversification. Their findings were in line with a possible trade-off between international diversification and risk.

The particular structure of our data sample is somewhat related to those of Chollete et al. [7, 8]. One part of the sample in these contributions consists of developed stock markets and the remaining part of emerging markets, as in our paper. Therefore the approach and the conclusions of Chollete et al. [7, 8] are the basis for the setup of our paper. However, we estimate one model connecting all the markets, while every region is modelled separately in [7] and pair dependences in [8].

The main goal of this article is to present the construction and examination by *VaR* and *ES* of the optimal investment strategy in selected European markets (both well developed and emerging markets as well). The *ex post* analysis provides us with a series of optimal portfolios; this series is analyzed. This is done in order to verify whether the extent of market capitalization or the maturity of a market has an influence on the optimal portfolio. Moreover, the question of whether composition of the optimal portfolio depends on the global financial situation (bear or bull markets) is explored. In addition, we examine the effect of the index size on its skewness, fat-tailedness, the strength of dependences among indices, the probability of staying in a particular regime versus shifting. We also check the dynamics of *VaR* and *ES*. We conduct analysis combining regime switching models and the modern vine-copula approach. In this way, we allow the dependence structure to change over time in a multivariate framework.

The remainder of the paper is organized as follows. A description of the mathematical definitions of Value at Risk and Expected Shortfall, and the model setup is given in Section 2. In this section, there is also an overview of features of the model and an estimation of the parameters. In Section 3, extensive empirical results are presented. Conclusions summarizing the most significant findings are given in the last section.

## 2. Model and methodology

The proposed model is motivated by the properties of multivariate financial data. Asymmetries in margins and dependences are taken into consideration, the dynamic in margins is included and the changeability of financial situation is described by

a regime-switching latent variable. The model is based on the concept of a copula function. Copulas are attractive and efficient tools for describing the dependence structure between financial data. Moreover, one can estimate models describing margins and the copula describing relationships separately.

### 2.1. Marginal model

We have used the ARMA model to describe a margin's mean and the univariate GARCH model to describe its variance. Numerous modifications and generalizations of the GARCH model have been proposed. There is no point in employing an expanded model in our approach. These models have been mostly used within the approach of multivariate GARCH models to describe margins. In this case, a certain conditional distribution of a margin is needed. Using a copula function to describe the dependence structure, there is no restriction on margin distributions. In particular, the AR(1)-GARCH(1,1) model seems to be the right one in the case under study. Therefore, for  $i = 1, \dots, n$ , where  $n$  is the dimension of the data, the  $i$ -th margin in time  $t$ , denoted by  $y_{i,t}$ , is described by:

$$y_{i,t} = a_i y_{i,t-1} + \sqrt{h_{i,t}} \varepsilon_{i,t}$$

where  $a_i$  is an autoregressive term,  $h_{i,t}$  denotes the variance process and  $\varepsilon_{i,t}$  is an i.i.d. process such that  $\varepsilon_i$  is independent of  $\varepsilon_j$  for  $i \neq j$ . The variance  $h_{i,t}$  is described by:

$$h_{i,t} = \omega_i + \alpha_i h_{i,t-1} + \beta_i \varepsilon_{i,t-1}^2$$

and

$$\varepsilon_{i,t} \sim SS_t(\nu_i, \lambda_i)$$

where  $SS_t$  denotes a skewed-Student's- $t$  distribution. The distribution,  $g(x|\nu, \xi)$ , where  $\nu > 2$  is the degree of freedom and  $\xi \in \mathbb{R}_+$  is a shape parameter, is given by:

$$g(x|\nu, \lambda) = \frac{2}{\xi + \xi^{-1}} (f(\xi z) H(-z)) + f(\xi^{-1} z) H(z)$$

where  $f$  is the standard Student's- $t$  distribution density and  $H$  – the Heaviside step function.

The economic meaning of the  $\nu$  parameter is heavy-tailedness, the smaller  $\nu$  the heavier the tail is. A higher value than unity of the  $\lambda$  parameter corresponds to a right-

skewed density, and a lower value corresponds to a left-skewed one. The latter is expected, which means that extreme negative values appear with a higher probability than extreme positive values.

Other distributions for describing the error term have been proposed, for instance, generalized error distribution (*ged*). However, we find no relevant advantages of the *ged* over the *SSt* distribution and decide not to extend the model unless necessary.

## 2.2. Copulas

Following the results presented in the works mentioned in Section 1, we assume two states of the market. The bull phase, characterized by weak interrelations, is described by a Gaussian copula. The variables connected by a Gaussian copula are symmetrically related and tail-independent.

The construction of multidimensional copulas is difficult; standard copulas with the dimension higher than 2 are elliptical and Archimedean copulas. However, neither an elliptical copula nor an Archimedean one describes the bear phase properly. An elliptical copula corresponds to a symmetric dependence and an Archimedean copula links every pair in the data in the same way. Conversely, there is a developed theory of bivariate copulas, so that, it is possible to describe the dependence of every pair in any data analyzed properly. However, the complete multivariate dependence structure cannot be determined by the pair relations. These remarks lead us to the vine copulas proposed in [1], which were referred to in the first part of our paper. A vine copula construction is based on decomposition into a cascade of bivariate copulas and conditional copulas. Any decomposition is not unique, basic two families of vine copulas have been put forwarded, canonical (C-vine) and drawable (D-vine) copulas. Moreover, the order of the data affects the form of the vine copula. We have chosen a C-vine with an order corresponding to the strength of the relation, in our case measured by Kendall tau. This choice was justified in [1]. The chosen vine copula corresponds to the following decomposition of an  $n$ -dimensional distribution  $f$ :

$$f(x_1, \dots, x_n) = f(x_1) \prod_{i=2}^n f(x_i | x_1, \dots, x_{i-1}), \quad x_1, \dots, x_n \in \mathbb{R} \quad (1)$$

For the sake of clarity, we introduce the following notation:

$$\begin{aligned} F_{i|j} &:= F_{i|1, \dots, j-1}, & f(x_i | x_{-j}) &= f(x_i | x_1, \dots, x_{j-1}) \\ F_{i|1} &:= F_i, & f(x_i | x_{-1}) &= f(x_i) \end{aligned}$$

where  $i, j \in \mathbb{N}$  are any indices,  $F$  and  $f$  are conditional functions.

The density of a C-vine copula, denoted by  $c$ , for  $y_1, \dots, y_n \in \mathbb{R}$  and marginal cumulative distribution functions  $F_1, \dots, F_n$  is given by:

$$c(y_1, \dots, y_n) = \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{j,j+i|j} \left( F_{j|j} (y_j | y_{-j}), F_{j+i|j} (y_{j+i} | y_{-j}) \right) \quad (2)$$

where  $c_{j,j+i|j}$  is the density of the bivariate copula which connects the  $j$ -th and the  $(j+i)$ -th variables conditionally on variables  $1, \dots, j-1$ . For  $j = 1, \dots, n$  and  $k = j+1, \dots, n$ , the conditional cumulative distribution function  $F_{k|j}(y_k | y_{-j})$  is given by:

$$\begin{aligned} F_{k|j} (y_k | y_1, \dots, y_j) &= \frac{\partial C_{j,k|j} \left( F_{k|j} (y_k | y_{-j}), F_{j|j} (y_j | y_{-j}) \right)}{\partial F_{j|j} (y_j | y_{-j})} \\ &= \int_0^{F_{k|j}(y_k | y_{-j})} c_{j,k|j} \left( x, F_{j|j} (y_j | y_{-j}) \right) dx \end{aligned} \quad (3)$$

where  $C_{j,k|j}$  is a copula function with the distribution  $c_{j,k|j}$ .

Instead of there being no suitable copula to describe an asymmetric multivariate data, we have an almost unlimited class of such copulas by involving vine copulas. Countless classes of bivariate copulas have been proposed, and it is possible to describe every conditional copula by whichever bivariate copula. Elliptical copulas correspond to elliptical distributions, making them a natural choice. The class of Archimedean copulas is successful because of its convenient form and flexibility. In the bivariate case, it is possible to model specific properties by an Archimedean copula. In the case of financial data, a model for lower tail dependence is needed. Taking these needs into consideration, we have arbitrarily chosen Gaussian, Clayton, Frank and rotated Gumbel copulas. The bivariate Gaussian copula class is one parameter class, as are the chosen Archimedean copulas. Clayton and rotated Gumbel copulas are lower tail dependent and the Gaussian and Frank copula are tail independent. There are

$d = \binom{n}{2}$  bivariate conditional copulas to determine in the C-vine structure and 4 possible choices for each one, so we have  $4^d$  possible choices and  $d$  real parameters for each one.

### 2.3. Regime switching model

We now introduce the two state Markov process  $\{S_t\}_{t=1}^T$ , which describes the dual nature of financial data. For  $t = 1, \dots, T$ , the process  $S_t$  takes values in  $\{1, 2\}$ , the states

being called regimes with the first regime corresponding to the symmetric phase and the second one to the asymmetric phase of the market. For  $i, j = 1, 2$ , we denote the corresponding transition probability by  $p_{ij}$ , so that:

$$p_{ij} := P(S_{t+1} = j | S_t = i), \quad t = 1, \dots, T \quad (4)$$

Transition probabilities form the transition matrix  $\mathbf{P}$ :

$$\mathbf{P} := \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{pmatrix} \quad (5)$$

Therefore, the conditional (on being in the regime  $j = 1, 2$ ) density  $f$  of the multivariate variable at time  $t$ , denoted by  $Y_t = (y_{1,t}, \dots, y_{n,t})$ , is given by:

$$f(Y_t | Y_{t-1}, S_t = j) = c^j(F_1(y_{1,t}), \dots, F_n(y_{n,t}); \theta_c^j) \prod_{i=1}^n f_i(y_{i,t}; \theta_i) \quad (6)$$

where  $c^j$  is the copula in the regime  $j$ , with the set of parameters  $\theta_c^j$ , and  $f_i$  being the density of the marginal distribution  $y_i$ , with the set of parameters  $\theta_i$ , for  $i = 1, \dots, n$ . Following the logic of Section 1, we determine that  $c^1$  is the Gaussian copula and  $c^2$  is the C-vine copula. Note that we assume marginal distributions to be independent on the regime switching model.

## 2.4. Estimation procedure

The estimation is based on a maximum likelihood estimation and is performed in two steps. This procedure was introduced by Hamilton [11] and widely analyzed in [12]. Formally, for the sample  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_T)$ , the log likelihood function is defined by:

$$L(\mathbf{Y}; \theta^m, \theta^d) = \sum_{t=1}^T \ln f(Y_t | Y_{t-1}; \theta^m, \theta^d) \quad (7)$$

and it decomposes to  $L_m$  which corresponds to the marginal part, and  $L_d$  which corresponds to the dependence structure. Therefore:

$$L(\mathbf{Y}; \theta^m, \theta^d) = L_m(\mathbf{Y}; \theta^m) + L_d(\mathbf{Y}; \delta, \theta^d) \quad (8)$$

where

$$L_m(\mathbf{Y}; \boldsymbol{\theta}^m) = \sum_{t=1}^T \sum_{i=1}^n \ln f_i(y_{i,t} | \boldsymbol{\theta}_i) \quad (9)$$

$$L_d(\mathbf{Y}; \boldsymbol{\theta}^m, \boldsymbol{\theta}^d) = \sum_{t=1}^T \ln \left( c \left( F_1(y_{1,t} | Y_{1,t-1}; \boldsymbol{\theta}_1), \dots, F_n(y_{n,t} | Y_{n,t-1}; \boldsymbol{\theta}_n); \boldsymbol{\theta}^d \right) \right) \quad (10)$$

and  $\boldsymbol{\theta}^m = (\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_n)$  is a set of parameters of marginal models, and  $\boldsymbol{\theta}^d = (\boldsymbol{\theta}_c^1, \boldsymbol{\theta}_c^2, p_{11}, p_{22})$  is the set containing the parameters of the copulas and the regime switching process.

It is not feasible to perform the estimation in one step because the number of parameters in the model is substantial and C-vine copulas are strongly non-linear, so we have maximized the  $L_m$  function, using a procedure which is a standard estimation of  $n$  univariate GARCH models separately, and which is available using any statistical software.

Secondly, the maximization of the  $L_d$  function is performed. However, the form of copula  $c$  in Eq. (10), at time  $t$ , is not known, since the process  $S_t$  is latent. We have therefore used the Hamilton filter to estimate the regime process. For the transition matrix  $\mathbf{P}$  given by (5), we define:

$$\hat{\xi}_{t|t} = \frac{\hat{\xi}_{t|t-1} \odot \boldsymbol{\eta}_t}{\mathbf{1}^T (\hat{\xi}_{t|t-1} \odot \boldsymbol{\eta}_t)} \quad (11)$$

$$\hat{\xi}_{t+1|t} = \mathbf{P}^T \hat{\xi}_{t|t} \quad (12)$$

where  $\hat{\xi}_{t|t} = P[S_t = j | Y_t; \boldsymbol{\theta}]$  and  $\hat{\xi}_{t+1|t} = P[S_{t+1} = j | Y_t; \boldsymbol{\theta}]$ , the Hadamard's multiplication denoted by  $\odot$  means multiplication coordinate by coordinate. The vector of copula densities is denoted by  $\boldsymbol{\eta}_t$ , hence,

$$\boldsymbol{\eta}_t = \begin{bmatrix} c^1(F_1(y_{1,t}; \boldsymbol{\theta}_1), \dots, F_n(y_{n,t}; \boldsymbol{\theta}_n); \boldsymbol{\theta}_c^1) \\ c^2(F_1(y_{1,t}; \boldsymbol{\theta}_1), \dots, F_n(y_{n,t}; \boldsymbol{\theta}_n); \boldsymbol{\theta}_c^2) \end{bmatrix} \quad (13)$$

The log likelihood function defined by (10) for the observed data can be rewritten as:

$$L_d(\mathbf{Y}; \boldsymbol{\theta}^m, \boldsymbol{\theta}^d) = \sum_{t=1}^T \ln \left( \mathbf{1}^T (\hat{\xi}_{t|t-1} \odot \boldsymbol{\eta}_t) \right) \quad (14)$$

where the initial value  $\hat{\xi}_{\eta|0}$  is the limit probability vector:

$$\hat{\xi}_{\eta|0} = \begin{bmatrix} \frac{1-p_{22}}{2-p_{11}-p_{22}} \\ \frac{1-p_{11}}{2-p_{11}-p_{22}} \end{bmatrix} \quad (15)$$

The standard errors of parameters corresponding to the dependence structure may be calculated using the Outer Product of Gradient matrices. However, we re-estimate the model, and it would be difficult to control the significance of the parameters during the whole procedure.

## 2.5. Portfolio optimization

In the paper, the portfolio optimization problem involves procedures to provide the portfolio with the minimal risk in the class of portfolios with a given expected return. We will construct two series of portfolios minimizing the *VaR* and minimizing the *ES*.

For the financial position  $X$  and  $\lambda \in (0, 1)$ , we define *VaR* at level  $\lambda$  as:

$$VaR_{\lambda}(X) := \inf \{m \mid P[X + m < 0] \leq \lambda\} \quad (16)$$

The  $VaR_{\lambda}(X)$  value is the smallest amount of capital which, if it added to position  $X$  and invested in a risk-free asset, keeps the probability of a negative outcome below a fixed level  $\lambda$ . In other words,  $VaR_{\lambda}(X)$  is the  $(1 - \lambda)$  quantile of the variable  $(-X)$ . The *VaR* is not a convex risk measure, which means that portfolio diversification may not decrease the *VaR*. The *ES* is a convex risk measure with a similar interpretation to *VaR*. Moreover, the *ES* is sensitive to the size of extreme negative outcomes, and the *VaR* is not.

For financial position  $X$  and  $\lambda \in (0, 1)$ , we define *ES* at level  $\lambda$  as:

$$ES_{\lambda}(X) := E[VaR_{\alpha}(X) \mid \alpha \leq \lambda] \quad (17)$$

Let the  $n$ -dimensional vector process  $\{y_t\}$  be represented by the copula-switching model. The portfolio  $p_t^{VaR}$  is obtained as a result of the following optimization problem:

$$\begin{aligned}
\sum_{i=1}^n p_{i,t}^{VaR} &= 1 \\
E\left(\sum_{i=1}^n p_{i,t}^{VaR} y_{i,(t+1)|t}\right) &= r \\
VaR_{\alpha}\left(\sum_{i=1}^n p_{i,t}^{VaR} y_{i,(t+1)|t}\right) &\longrightarrow \min!
\end{aligned} \tag{18}$$

where  $p_{i,t}^{VaR}$  is the  $i$ -th coordinate of the vector  $\mathbf{p}_t^{VaR}$ , and  $y_{i,(t+1)|t}$  denotes the conditional distribution of the  $i$ -th asset return, for  $i = 1, \dots, N$ . The  $\mathbf{p}_t^{VaR}$  vector is an  $n$ -dimensional time series which represents an investment with expected return  $r$  at time  $t + 1$  and minimal  $VaR$ .

Analogously, the portfolio  $\mathbf{p}_t^{ES}$  is built as result of the following optimization problem:

$$\begin{aligned}
\sum_{i=1}^n p_{i,t}^{ES} &= 1 \\
E\left(\sum_{i=1}^n p_{i,t}^{ES} y_{i,(t+1)|t}\right) &= r \\
ES_{\alpha}\left(\sum_{i=1}^n p_{i,t}^{ES} y_{i,(t+1)|t}\right) &\longrightarrow \min!
\end{aligned} \tag{19}$$

where  $p_{i,t}^{ES}$  are coordinates of the vector  $\mathbf{p}_t^{ES}$ . It is the  $N$ -dimensional time series, which represents the investment at time  $t$  with expected return  $r$  at time  $t + 1$ , and minimal  $ES$ .

### 3. Empirical results

In this section, we describe the data and present basic descriptive statistics and empirical relations. We briefly discuss the result of the copula switching model estimation for the entire data set. Finally, we describe the estimation procedure for the *ex post* analysis and analyse the outcome of the two portfolio optimization procedures.



The highest dependences among the three biggest stock markets (the British, the French and the German) are visible. The Turkish index is the least related to the others.

### 3.2. Results of estimation

In this section, we present the estimation results obtained by conducting the procedure presented in Section 2. We perform the procedure using all available data, so that the result may differ from the results in the *ex post* analysis. Table 3 presents the results of marginal estimates.

Table 3. Marginal estimates

Stock market	$ar1$	$\alpha$	$\beta$	$\zeta$	$\nu$
CAC	-0.0537	0.0831	0.9107	0.9143	10.3713
DAX	-0.0335	0.0856	0.9085	0.9120	10.2589
FTSE	-0.0527	0.0961	0.8950	0.9192	10.8834
ATX	0.0307	0.0817	0.9098	0.9422	7.2649
PX	0.0247	0.1049	0.8784	0.9839	7.7866
BUX	0.0240	0.0761	0.9065	0.9694	7.9148
BIST	0.0289	0.1140	0.8813	0.9773	5.9070

Almost every parameter in all seven marginal results of estimation is significant with the 10% significance level (except for the  $ar1$  parameter for the PX and the BUX indices). We therefore decide not to re-estimate models with the exclusion of insignificant parameters, and for that reason we do not include the standard errors of the parameters. Moreover, we ensure that properties like skewness or fat-tailness are taken into account. The  $\alpha$  and the  $\beta$  parameters are similarly framed in every series, about 90% of the standard deviation value comes from the previous value and about 10% comes from the realization of the previous error term. As has been expected, every series is left-skewed. The bigger the index, the more skewed (FTSE, CAC and DAX), with the least skewed being the PX index. The degree of freedom parameter  $\nu$  is smallest for the BIST index (at 5.9), it is between 10 and 11 for FTSE, CAC and DAX, and between 7 and 8 for ATX, PX and BUX. The three highest indices prove to be less fat-tailed than the others while the BIST index proved to be more fat-tailed than the others. We decided not to replace the  $t$  distribution with the normal for any series.

In Tables 4 and 5, we present the results of estimation of both copulas. Table 4 contains correlation coefficients for the symmetric Gaussian copula in the first regime. In this regime, we observe positive correlations for all pairs. The dependence is strongest among the three biggest indices (FTSE, CAC and DAX). Furthermore, the correlations of the Turkish index are significantly lower than the correlations among the others. The observation implies that the general dependence of the BIST on other indices lies in the asymmetric regime.

The choice of C-vine structure and order is motivated by the empirical Kendall tau presented in Table 2. A canonical structure of vine copulas is recommended where there is a variable which has the strongest relation to the others. In our case, the pivotal variable is the CAC. The order for the remaining variables is adopted in accordance with the values of the Kendall tau. For more details on choosing a vine structure see [1, 7, 10].

Table 4. The Gaussian copula estimates

Stock market	CAC	DAX	FTSE	ATX	PX	BUX	BIST
CAC	1.0000	0.7976	0.7472	0.4865	0.3705	0.3642	0.0772
DAX	0.7976	1.0000	0.6281	0.4184	0.2775	0.3027	0.0823
FTSE	0.7472	0.6281	1.0000	0.4312	0.3395	0.3110	0.1244
ATX	0.4865	0.4184	0.4312	1.0000	0.4624	0.3988	0.0925
PX	0.3705	0.2775	0.3395	0.4624	1.0000	0.4484	0.1545
BUX	0.3642	0.3027	0.3110	0.3988	0.4484	1.0000	0.1319
BIST	0.0772	0.0823	0.1244	0.0925	0.1545	0.1319	1.0000

Table 5. The C-vine copula estimates<sup>1</sup>

Stock market	DAX	FTSE	ATX	PX	BUX	BIST
CAC	rGumbel 3.1141	rGumbel 2.2266	rGumbel 1.6729	Frank 3.8326	rGumbel 1.4442	rGumbel 1.3085
DAX		Gauss 0.1621	Frank 0.8379	Frank 0.7171	Frank 0.6190	Frank 0.7414
FTSE			Frank 0.6599	Clayton 0.0793	Gauss 0.0871	Frank 0.9228
ATX				Gauss 0.3120	Gauss 0.2412	Frank 1.0676
PX					Gauss 0.3026	Gauss 0.2224
BUX						Frank 1.1191

<sup>1</sup>This table presents families and parameters of estimated bivariate copulas forming the C-vine copula in the second regime. The first row consists of the copulas linking CAC with the others; the second – copulas linking DAX with the others conditional on CAC; the third – the copulas linking FTSE with the others conditional on CAC and DAX; the fourth – the copulas linking ATX with PX, BUX and BIST conditional on the others, the fifth – the copulas linking PX with BUX and with BIST conditional on the others and the last – the copula linking BUX with BIST conditional on the others.

The interpretation of the Gaussian copula estimates presented in Table 4 is clear. In order to clarify the results of estimation for the C-vine copula presented in Table 5,

we evaluate the strength of connections and tail dependences for this copula. In Table 6, we present the Kendall tau of variables linked by the estimated C-vine copula, the values being calculated as the realized Kendall tau for a simulated sample 1 000 000 in size. Analogously, in Table 7 we present lower tail dependences of variables linked by the estimated C-vine copula. Note that upper tail dependences equal zero for all pairs, because of the choice of pair copulas. These results show lower fat-tailedness and asymmetry in the dependence in the second regime.

Table 6. The C-vine Kendall  $\tau$ 

Stock market	CAC	DAX	FTSE	ATX	PX	BUX	BIST
CAC	1.0000	0.7953	0.6425	0.5699	0.4352	0.4514	0.3882
DAX		1.0000	0.6174	0.5508	0.4157	0.4363	0.3827
FTSE			1.0000	0.4881	0.3675	0.3739	0.3913
ATX				1.0000	0.4821	0.4158	0.3895
PX					1.0000	0.4412	0.3469
BUX						1.0000	0.3349
BIST							1.0000

Not all parameters of the pair copulas in lower levels of the C-vine structure are significant. However, the hypothesis of independence between any two variables is strongly rejected, and as part in our analysis, we re-estimate the model repeatedly. Hence, we include all the estimated terms during the procedure.

Table 7. The C-vine lower tail dependence

Stock market	CAC	DAX	FTSE	ATX	PX	BUX	BIST
CAC	1.00	0.54	0.45	0.42	0.00	0.28	0.21
DAX		1.00	0.43	0.39	0.04	0.30	0.21
FTSE			1.00	0.33	0.07	0.25	0.20
ATX				1.00	0.07	0.29	0.19
PX					1.00	0.08	0.09
BUX						1.00	0.15
BIST							1.00

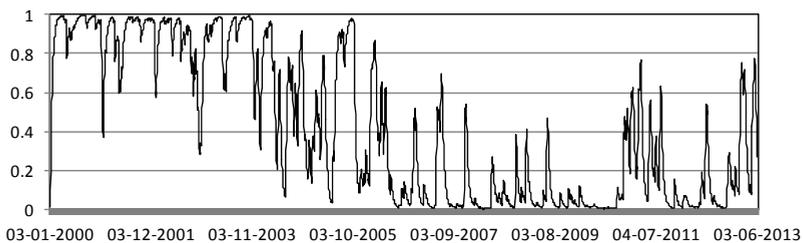


Fig. 1. Smoothed probability of low dependence regime

Transition probabilities  $p_{11}$  and  $p_{22}$  are 0.9664 and 0.9895, respectively. That is why, staying in a particular regime is much more probable than shifting. However, Fig. 1 presents the estimated probability of being in the first regime, which is rather variable. It has been observed that in the period 2006–2010, the market was mostly in the asymmetric regime, in other periods it was more variable, although, in the period 2000–2005 there was a dominance of the symmetric regime, and in the period 2011–2013 a dominance of the asymmetric regime.

### 3.3. Risks and optimal portfolios

We construct optimal portfolios using the procedures (18) and (19) presented in Section 2.5. We conduct the *ex post* analysis as follows. Firstly, we estimated the

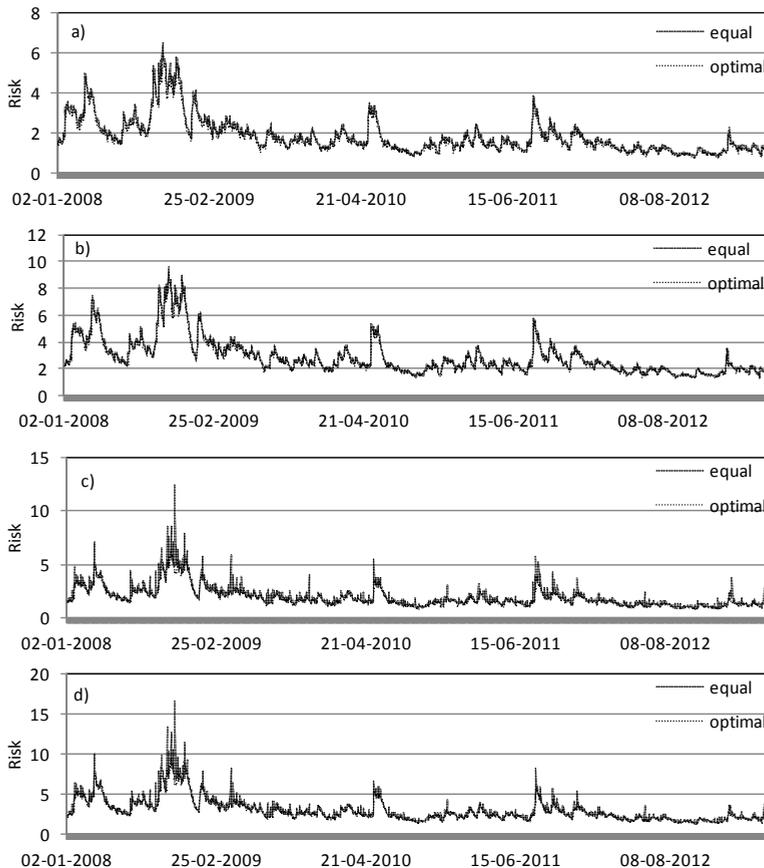


Fig. 2.  $VaR_{0.1}$  (a),  $ES_{0.1}$  (b),  $VaR_{0.5}$  (c) and  $ES_{0.5}$  (d) of the optimal and the equally weighted portfolio

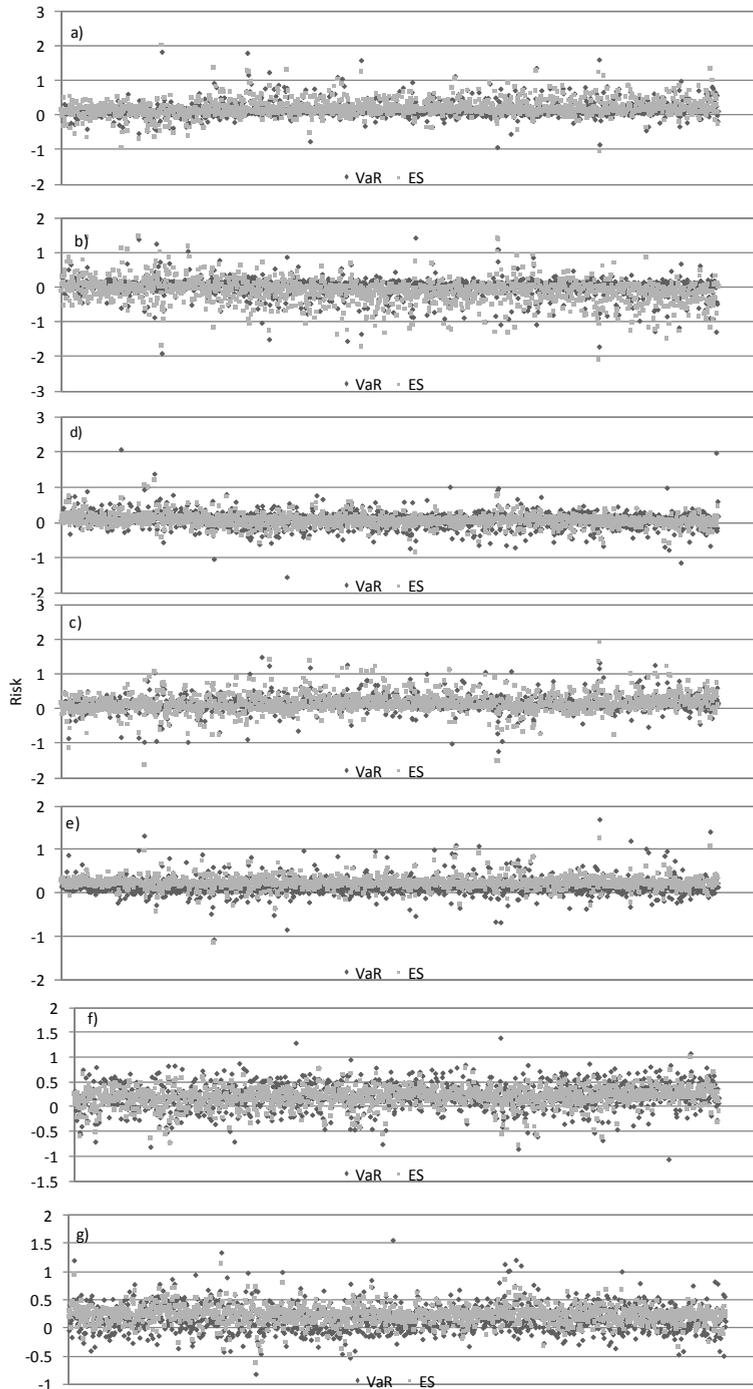


Fig. 3. Shares of indices in optimal portfolios: a) FTSE; b) CAC; c) DAX; d) ATX; e) PX; f) BUX; g) BIST

model (margins and the copula-regime-switching) using the period of January 2000 to December 2007 (observation No. 2086). Secondly, we simulated 100 000 observations from the estimated switching copula model, taking into account the period up to the time point  $t = 2086$ . Then, the simulations were transformed by the corresponding estimated marginal distributions. The set of 100 000 vectors is considered as the forecast for the time  $(t + 1)$ , conditional on history up to the time  $t$ . The procedure is repeated for all further time points, re-estimating parameters of the model every 50 observations and at each estimation using a historical period of the same length (2086 observations).

In the analysis, we fixed the levels of  $VaR$  and  $ES$  at  $\lambda = 10\%$  and  $\lambda = 5\%$ . Optimal portfolios are calculated using Eqs. (18) and (19); the expected daily logarithmic return is assumed to be not lower than the expected return of an equally weighted portfolio. We see that  $VaR$  and  $ES$  have similar dynamics in the analyzed period (Fig. 2). The difference between the optimal and the equally weighted portfolios is more significant in the case of  $ES$ . Additionally, the expected value of the return defined in Eq. (20) is significantly higher in the case of an optimal portfolio. The standard Kupiec test's  $p$ -value equals 0.0677, thus the model estimates  $VaR$  properly.

Figure 3 presents the division process of the  $VaR$  and the  $ES$  optimal portfolios. The optimal portfolios are highly volatile, and it is difficult to find any regularity or relations with the current regime. It can be seen, that DAX and PX are on the whole unattractive, recommended to sell. Conversely, FTSE, ATX, BUX and BIST are mostly recommended to buy.

A standard method for the visualization of a measure of risk, under an assumed model, is to draw on of the efficient frontier line. The efficient frontier for a given measure of risk is the curve showing the minimal risk of a portfolio which exhibits given expected returns. We have created efficient frontiers to show the differences between the two estimated regimes and the two analysed risk measures. Knowing the distribution of the future outcome of available assets, the efficient frontier shows the risk to be taken by an investor for an increase in the expected return of an investment.

Clearly, it is impossible to present efficient frontiers at every time. Figure 4 is created as follows: using the whole sample, the model is estimated. The simulation of 100 000 vectors under the assumption of the first regime conditional on history is performed. Intuitively this is a one step forecast. Similarly for the second regime. These 100 000 simulations have been taken as the distribution of the random vector under the assumption of being in a certain regime.

Figure 4 illustrates two important relations. By definition, the  $ES$  is higher for all expected returns and for both regimes. Moreover, the increase in the  $ES$  is much faster than in the  $VaR$ , through an increase in expected returns in the C-vine case. The influence of the regime state on both risk measures is intense, and values and slopes are significantly higher in the asymmetric regime for both risk measures.

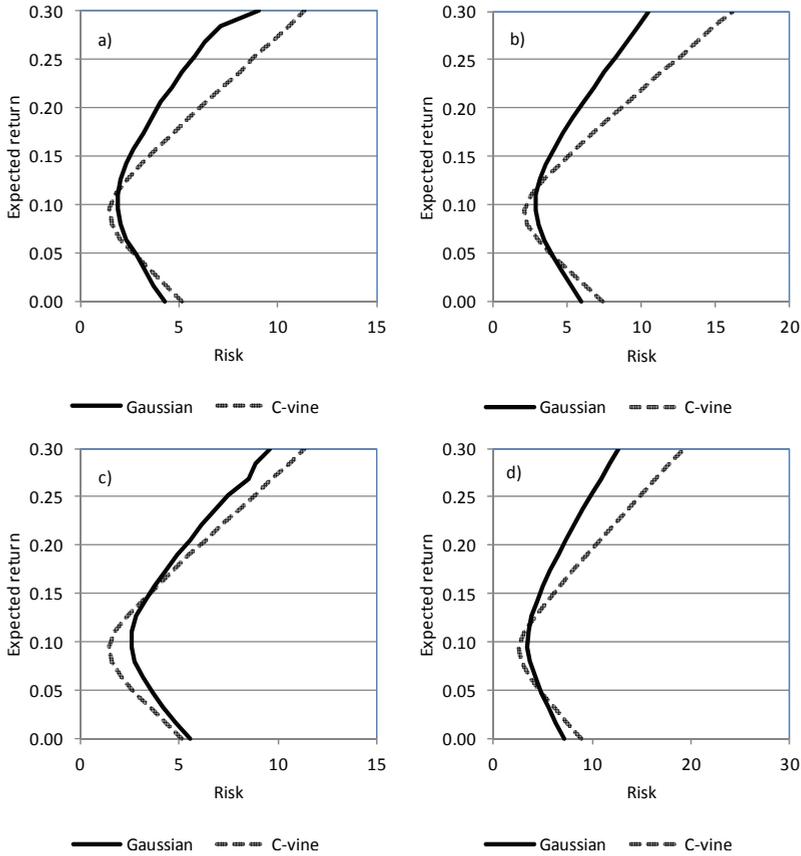


Fig. 4. Efficient frontiers of the  $VaR_{0.1}$  (a),  $ES_{0.1}$  (b),  $VaR_{0.5}$  (c) and  $ES_{0.5}$  (d)

Additionally, the irregular shape of an efficient frontier in the  $VaR$  cases strongly suggests the superiority of the  $ES$  over the  $VaR$ . Irregularities in the shape of efficient frontiers for C-vines could be caused by properties of the  $VaR$  function itself. However, irregular shapes in Gaussian regimes are caused by the inaccuracy of the Monte Carlo method. The  $ES$  calculation is more precise, because it uses all simulated observations under an assumed level. Conversely, the  $VaR$  calculation is based on one particular value in the simulated sample.

## 4. Conclusions

An analysis of the dependence structure among selected European markets is conducted in this paper. The model is chosen to include important features of the financial data: asymmetry, fat-tailedness, variability in time. The chosen model is complex,

based on recent empirical and theoretical studies. Applying the concept of copula functions, we are able to model margins irrespective of the dependence structure. One-dimensional modelling has been widely studied, and we choose one of the suitable models, namely the AR(1)-GARCH(1,1) model with a skewed conditional distribution. The dependence structure of multi-dimensional data is complex. We estimate a regime switching model based on copulas including asymmetric and fat-tailed copulas. The Markov regime switching latent variable models the two-state nature of a market (bull or bear market). Including certain bivariate Archimedean copulas in the C-vine construction, we are able to elucidate asymmetric and fat-tailed dependences between variables. We see fat-tailedness only in the lower tail, and the theoretical lower tail dependence for the estimated C-vine copula in an asymmetric regime appeared to be positive for the majority of pairs.

All analyzed indices (FTSE, CAC, DAX, ATX, PX, BUX and BIST) are left-skewed and fat-tailed. Additionally, bigger indices appeared to be more skewed and less fat-tailed. Generally, FTSE, CAC and DAX show similar properties, and ATX, PX and BUX appear to have similar properties, and BIST is different from the rest. These relations suggest that the size of a market has an influence on its properties, the dissimilarity of the Turkish market suggesting the influence of the geographical aspect. It is also suggested that the maturity of a market is insignificant in the analysis.

The dependence structure has several important properties. All returns are positively correlated and the strongest correlation is observed among FTSE, CAC and DAX. Furthermore the relations between the indices are not constant in time. This variability is described by the regime switching model with two regimes. The first regime corresponds to a symmetric period and is described by a Gaussian copula, the second regime, described by a vine copula, corresponds to an asymmetric period. The asymmetric regime describes a strong and asymmetric dependence. Extreme negative outcomes are more probable in this regime. The persistence of both regimes is high. Moreover, the asymmetric regime appears to be more persistent, and the probability of shifting from a symmetric regime to an asymmetric one is higher than the reverse. Proper evaluation of the dependence structure is particularly important in risk management. Using the presented model, we discuss Value at Risk and Expected Shortfall in European markets. Both measures show similar dynamics. We construct optimal portfolios with respect to the risk measures and compare them with equally weighted portfolios. The risk of optimal portfolios proved to be lower than the risk of equally weighted portfolios by 0.14 percentage points on average. The optimal portfolio composition is highly variable. The cost of such an investment strategy is an intense change in portfolio composition. This requires almost complete decomposition of a portfolio every day. The results show that the correct evaluation of the current state is decisive with respect to risk calculation.

The values of these risk measures appeared to have the same dynamics. This means that a change in the VaR of a portfolio takes place along with a proportional

change in the  $ES$  value, so that  $VaR$  and  $ES$  studies lead to similar conclusions.  $ES$  is recommended due to its theoretical properties such as convexity and homogeneity. Apart from that, additional analysis strongly suggests that  $VaR$  is calculated less precisely than  $ES$ .

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### References

- [1] AAS K., CZADO C., FRIGESSI A., BAKKEN H., *Pair-copula construction of multiple dependence*, Insurance: Mathematics, Economics, 2009, 44, 182–198.
- [2] BEDFORD T., COOKE R., *Probability density decomposition for conditionally dependent random variables modeled by vines*, Annals of Mathematics, Artificial Intelligence, 2001, 32, 245–68.
- [3] BEDFORD T., COOKE R., *Vines – a new graphical model for dependent random variables*, Annals of Statistics, 2002, 30, 1031–1068.
- [4] BERG D., *Copula goodness-of-fit testing: an overview, power comparison*, The European Journal of Finance, 2009, 15, 1466–4364.
- [5] BILLIO N., PELIZZON L., *Value at risk: a multivariate switching regime approach*, Journal of Empirical Finance, 2000, 7, 531–554.
- [6] CHAN N.-H., CHEN J., CHEN X., FAN Y., PENG L., *Statistical inference for multivariate residual copula of GARCH models*, Statistica Sinica, 2009, 19, 53–70.
- [7] CHOLLETE L., HEINEN A., VALDESOGO A., *Modeling international financial returns with a multivariate regime-switching copula*, Journal of Financial Econometrics, 2009, 7 (4), 437–480.
- [8] CHOLLETE L., PEÑA V., LUC C.-C., *International diversification. A copula approach*, Journal of Banking and Finance, 2011, 35 (2), 403–417.
- [9] CZADO C., *Pair-copula constructions of multivariate copulas*, [in:] P.E.A. Jaworski (Ed.), *Copula theory, its applications*, Lecture Notes in Statistics, 198, Springer-Verlag, Berlin 2010, 93–109.
- [10] CZADO C., SCHEPSMEIER U., MIN A., *Maximum likelihood estimation of mixed C-vines with application to exchange rates*, Statistical Modelling, 2012, 12 (3), 229–255.
- [11] HAMILTON J.D., *A new approach to the economic analysis of nonstationary time series, the business cycle*, Econometrica, 1989, 57, 357–384.
- [12] HAMILTON J.D., *Analysis of time series subject to changes in regime*, Journal of Econometrics, 1990, 45, 39–70.
- [13] JOE H., *Families of m-variate distributions with given margins,  $m(m - 1)/2$  bivariate dependence parameters*, [in:] L. Riischendorf, B. Schweizer, M.D. Taylor (Eds.), *Distributions with fixed marginals, related topics*, Hayward, CA, Inst. Math. Statist., 1996, 28, 120–141.
- [14] JOE H., *Multivariate models, dependence concepts*, Chapman & Hall, London 1997.
- [15] JOE H., *Asymptotic efficiency of the two-stage estimation method for copula-based models*, Journal of Multivariate Analysis, 2005, 94, 401–19.

- [16] JONDEAU E., ROCKINGER M., *The copula-GARCH model of conditional dependencies: an international stock market application*, Journal of International Money, Finance, 2006, 25, 827–853.
- [17] KAWATA R., KIJIMA M., *Value at risk in a market subject to regime switching*, Quant. Finan., 2007, 7, 609–619.
- [18] KUESTER K., MITTNIK S., PAOLELLA M., *Value at risk prediction: a comparison of alternative strategies*, Journal of Financial Econometrics, 2006, 4 (1), 53–89.
- [19] KUROWICKA D., COOKE R.M., *Uncertainty analysis with high dimensional dependence modelling*, Wiley, Chichester 2006.
- [20] MASHAL R., ZEEVI A., *Beyond correlation. Extreme co-movements between financial assets*, Technical Report, Columbia University, New York 2002.
- [21] NEFTCI S.N., *Value at risk calculations, extreme events, fat tail estimation*, Journal of Derivatives, 2000, 7 (3), 23–37.
- [22] NELSEN R.B., *An introduction to copulas*, 2nd Ed., Springer-Verlag, Berlin 2006.
- [23] NING C., *Extreme dependence of international stock market*, Technical Report, Ryerson University, Toronto 2008.
- [24] OKIMOTO T., *New evidence on asymmetric dependence structures in international equity markets*, Journal of Financial, Quantitative Analysis, 2008, 48 (3), 787–815.
- [25] PATTON A., *On the out-of-sample importance of skewness, asymmetric dependence for asset allocation*, Journal of Financial Econometrics, 2004, 2 (1), 1301–1368.
- [26] RODRIGUEZ J., *Measuring financial contagion: a copula approach*, Journal of Empirical Finance, 2007, 14, 401–423.
- [27] ROSENBERG J., SCHUERMANN T., *A general approach to integrated risk management with skewed, fat-tailed risks*, Journal of Financial Economics, 2006, 79, 569–614.

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